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Markov's Inequality on Locally Lipschitzian Compact Subsets of $\mathbb{R}^N$ in L^p-Spaces

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NOTATION

For any $x=(x_1,..., x_N)\in\mathbb{R}^N$ we set $x'=(x_1,..., x_{N-1})$, $\|x\|=(\sum_{i=1}^N |x_i|^2)^{1/2}$, $\|x'\|=(\sum_{i=1}^{N-1} |x_i|^2)^{1/2}$, $D_i=\partial/\partial x_i$ $(i=1,..., N)$. For any $e\in\mathbb{R}^N$, D_e denotes the derivation in the direction e: $(D_e f)(x)=\mathrm{Lim}_{h\to 0}(f(x+he)-f(x))/h\|e\|)$. Let $a\in\mathbb{R}^N$ and $r>0$; we set $B(a,r)=\{x\in\mathbb{R}^N/\|x-a\|<r\}$, $S(a,r)=\{x\in\mathbb{R}^N/\|x-a\|=r\}$. A real-valued function φ defined on $\mathbb{R}^N$ is said to be locally Lipschitzian if for any $a\in\mathbb{R}^N$, there exist a neighborhood V of a and a positive constant k such that for any x and y in V we have $|\varphi(x)-\varphi(y)|\leqslant k\|x-y\|$. (We say φ is a k-locally Lipschitzian function in V.)

Let Ω be an open set in $\mathbb{R}^N$ with boundary $\partial\Omega$. Ω is said to be Lipschitzian if for any $a\in\partial\Omega$ there exist r, real and positive, φ locally Lipschitzian, local coordinates $x_1,..., x_N$ with origin at a, such that if $x\in\Omega\cap B(a,r)$, then $x_N>\varphi(x')$ and if $x\in\partial\Omega\cap B(a,r)$ then, $x_N=\varphi(x')$.

We denote by H_n the set of polynomials in N variables, of total degree at most n, and by $L^p(E)$ the space of measurable functions satisfying

$$\|f\|_{L^p(E)}=\left[\int_E |f(x)|^p\,dx\right]^{1/p}<\infty \qquad (1\leqslant p<\infty),$$

$$\|f\|_{L^\infty(E)}=\operatorname*{ess\,sup}_{x\in E}|f(x)|<\infty \qquad (p=\infty).$$

THE MAIN RESULT

The aim of this paper is to prove the following statement:

THEOREM. *Let Ω be a bounded locally Lipschitzian open set in $\mathbb{R}^N$. Then*

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(i) *for every* $p \geqslant 1$ *there exists a constant* $C(\Omega, p)$ *such that for any* $n \in \mathbb{N}$ *and any* $P \in H_n$ *we have*

$$\|D_i P\|_{L^p(\bar{\Omega})} \leqslant C(\Omega, p)\, n^2 \|P\|_{L^p(\bar{\Omega})} \qquad (i = 1, \ldots, N), \tag{1}$$

(ii) *there exists a constant* $C(\Omega)$ *such that for any* $p \geqslant 1$, $C(\Omega, p) \leqslant C(\Omega)$.

This theorem provides a generalization of Markov's inequality in L^p in the case of several variables. For the case of an interval see [2]. The assumption that Ω is locally Lipschitzian is fundamental to get the exponent 2 in (1). A counterexample is given in [1]: if $\Omega = \{(x, y) \in \mathbb{R}^2 / 0 < x < 1; 0 < y < x^p; p > 1\}$ the optimal exponent is $2p$. Furthermore, 2 is obviously the sharp exponent when Ω is a hypercube.

Sketch of the Proof. For any $a \in \Omega$ we find a neighborhood V_a of a such that

$$\|D_i P\|_{L^p(\bar{\Omega} \cap V_a)} \leqslant C_a n^2 \|P\|_{L^p(\bar{\Omega})} \qquad (i = 1, \ldots, N), \tag{2}$$

where C_a is a positive constant depending only on p and Ω. Since it is closed and bounded, $\bar{\Omega}$ is a compact set and thus can be covered with a finite set of neighborhoods such as V_a, denoted by $V_{a_1}, \ldots, V_{a_k}$. Clearly,

$$\|D_i P\|_{L^p(\bar{\Omega})} \leqslant \sum_{j=1}^{k} \|D_i P\|_{L^p(\bar{\Omega} \cap V_{a_j})}.$$

From this we immediately deduce the theorem.

To prove (2) after a linear change of variable, we can use local orthonormal coordinates. Furthermore if $e_1, e_2, \ldots, e_N$ are N independent elements in $\mathbb{R}^N$, proving (2) is equivalent to proving that there exists a constant C'_a such that for $i = 1, \ldots, N$

$$\|D_{e_i} P\|_{L^p(\bar{\Omega} \cap V_a)} \leqslant C'_a n^2 \|P\|_{L^p(\bar{\Omega})}. \tag{3}$$

We shall prove the inequalities (3).

Markov's Inequality on an Interval in L^p-spaces

Proposition 1 [2]. *Let* $I = [-1, 1]$. *For any* $p \geqslant 1$ *there exists a constant* $C(p)$ *such that for any polynomial P of degree at most n*

$$\|P'\|_{L^p(I)} \leqslant C(p)\, n^2 \|P\|_{L^p(I)}.$$

Exponent 2 is sharp, as can be seen by taking $P = P_n^{(2,2)}$ (Jacobi's polynomials in the ultraspherical case). In [2, p. 735] this result is also given in a more accurate form

$$\|P'\|_{L^1(I)} \leqslant 2n[(n+1)^{n+1} n^{-n}] \, \|P\|_{L^1(I)}$$

and if $p > 1$,

$$\|P'\|_{L^p(I)} \leqslant 2n[(p-1)^{(1/p)-1}(np+1)^{n+(1/p)}(np-p+1)^{1-n-(1/p)}] \, \|P\|_{L^p(I)}.$$

It is easy to prove that the expressions between the square brackets are always less than $2e^2 n$ and therefore $C(p) < 4e^2$.

COROLLARY 1. *Let J be an interval with length l. For any $p \geqslant 1$ and any $P \in H_n$ we have*

$$\|P'\|_{L^p(J)} \leqslant 2C(p)\, l^{-1} n^2 \|P\|_{L^p(J)}, \tag{4}$$

$$\|P'\|_{L^p(J)} \leqslant 8e^2 l^{-1} n^2 \|P\|_{L^p(J)}. \tag{5}$$

PROOF OF THE THEOREM

We are led to consider two types of points in $\bar{\Omega}$: points in Ω and points in $\partial\Omega$.

Let a be in Ω. There exists $r > 0$ such that $V_a = \{x / |x_j - a_j| < r \ (j = 1,..., N)\}$ is included in Ω. Using (4), and P being interpreted as a function of x_i with the other variables held fixed, we get

$$\int_{a_i - r}^{a_i + r} |D_i P(x)|^p \, dx_i \leqslant C(p)^p \, r^{-p} n^{2p} \int_{a_i - r}^{a_i + r} |P(x)|^p \, dx_i.$$

Integrating both sides as functions of $x_1,..., x_{i-1}, x_{i+1},..., x_N$ yields

$$\begin{aligned}\|D_i P\|_{L^p(V_a)} &\leqslant C(p)\, r^{-1} n^2 \|P\|_{L^p(V_a)} \\ &\leqslant C(p)\, r^{-1} n^2 \|P\|_{L^p(\bar{\Omega})}.\end{aligned}$$

Let a be in $\partial\Omega$. First we give an outline of the method. We construct a neighborhood V_a of a such that $V_a \cap \bar{\Omega}$ is a union of parallel segments, i.e., a part of cylinder bounded by two regular surfaces. This is possible, due to the fact that Ω is locally Lipschitzian (a continuous frontier is not a sufficient assumption). Results of [2] are then applied to the individual intervals.

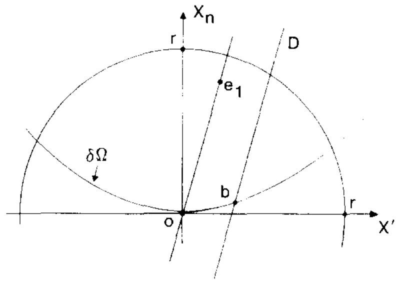

FIGURE 1

There exists an open ball $B(a, r)$, a k-locally Lipschitzian function φ, local coordinates with origin at a such that

$$\partial\Omega \cap B(a, r) = \{x \in B(a, r)/x_N = \varphi(x')\},$$

$$\Omega \cap B(a, r) = \{x \in B(a, r)/x_N > \varphi(x')\}.$$

From now on we use these coordinates; then $a = 0$, $\varphi(0) = 0$, $B(a, r) = B(0, r)$.

Let $e_1 = (1, 0, 0, ..., 0, 2k)$ be in $\mathbb{R}^N$, $b \in \partial\Omega \cap B(0, r)$ and $D = \{b + \lambda e_1/\lambda \in \mathbb{R}\}$ (see Fig. 1).

LEMMA 1. $D \cap \partial\Omega \cap B(0, r) = \{b\}$.

Proof. To prove the line D intersects $\partial\Omega$ in the ball only at the point b we establish that if $x = b + \lambda e_1$ with $\lambda \neq 0$ then $x_N \neq \varphi(x')$. Indeed for such an x, setting $b = (b', \varphi(b'))$ we have $x_1 = b_1$, $x_2 = b_2, ..., x_{N-1} = b_{N-1}$, $x_N = 2k(x_1 - b_1) + \varphi(b')$.

Since φ is k-locally Lipschitzian we have $|\varphi(x') - \varphi(b')| \leqslant k\|x' - b'\| = k|x_1 - b_1|$ and, using $x_N - \varphi(x') = 2k(x_1 - b_1) + \varphi(b') - \varphi(x')$, we get $|x_N - \varphi(x')| \geqslant k|x_1 - b_1|$ whence $x_N - \varphi(x') \neq 0$ if $x_1 \neq b_1$. ∎

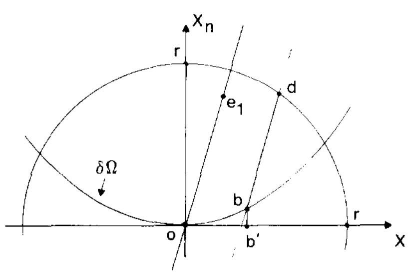

FIG. 2. The segment $[b, d]$.

Let $b' \in \mathbb{R}^{N-1}$ satisfying $\|b'\| \leqslant \frac{1}{2}r(1+k^2)^{-1/2}$. We have $|\varphi(b')| = |\varphi(b') - \varphi(0)| \leqslant k\|b'\| \leqslant \frac{1}{2}kr(1+k^2)^{-1/2}$. Thus $\|(b', \varphi(b'))\| < r/2$ and $b = (b', \varphi(b'))$ belongs to $B(0, r/2)$ and therefore to $B(0, r)$. The half-line $\{x \in \mathbb{R}^N / x = b + \lambda e_1 ; \lambda \geqslant 0\}$ intersects $S(0, r)$ at d (see Fig. 2).

LEMMA 2. *The segment $(b, d]$ is included in Ω and its length is greater than $r/2$.*

Proof. From Lemma 1, $(b, d]$ is included in Ω. Furthermore $\|d - b\| \geqslant \|d\| - \|b\| = r - \|b\|$. Then $b \in B(0, r/2)$ implies $\|d - b\| \geqslant r/2$. ▮

The line $\{\lambda e_1 / \lambda \in \mathbb{R}\}$ intersects $\partial\Omega \cap B(0, r)$ only at the origin. Therefore at the point $x = re_1/(2\|e_1\|)$ we have $x_N - \varphi(x') > 0$ and, at the point $-x$ we have $x_N - \varphi(x') < 0$. φ is a continuous function (all Lipschitzian functions are continuous) then $x_N - \varphi(x')$ remains positive in a neighborhood V_1 of $re_1/(2\|e_1\|)$ and negative in a neighborhood V_2 of $-re_1/(2\|e_1\|)$. Thus any segment with ends respectively in V_1 and V_2 intersects $\partial\Omega$. Whence using Lemma 2, we get:

COROLLARY 2. *For every α satisfying $0 < \alpha < \frac{1}{2}rk(1+k^2)^{-1/2}$ any line crossing $B(0, \alpha)$ and parallel to e_1 intersects $\partial\Omega \cap B(0, r)$ only at one point. Furthermore $D \cap \Omega \cap B(0, r)$ is a segment of length at least $r/2$ (see Fig. 3).*

We put $\sin\theta = (1 + 4k^2)^{-1/2}$, $\cos\theta = 2k(1 + 4k^2)^{-1/2}$ and we introduce new coordinates defined by

$$y_1 = x_1 \cos\theta - x_N \sin\theta,$$

$$y_N = x_1 \sin\theta + x_N \cos\theta,$$

$$y_i = x_i \qquad (i = 2, \ldots, N-1).$$

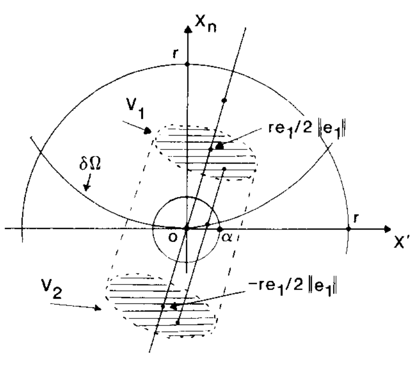

FIG. 3. How to find α.

With the y coordinates $S(0, r)$ is still defined by $\sum_{i=1}^{N} y_i^2 = r^2$. Let A and B be in $\mathbb{R}^N$. We denote by $(a_1, ..., a_N)$ and $(b_1, ..., b_N)$ the x coordinates of A and B respectively and by $(u_1, ..., u_N)$ and $(v_1, ..., v_N)$ the y coordinates of A and B, respectively.

LEMMA 3. *Assume* $|a_N - b_N| \leqslant k \|a' - b'\|$. *Then* $|u_N - v_N| \leqslant ((1 + 2k^2)/k)\|u' - v'\|$.

Proof. If $u_N \neq v_N$ we have

$$\frac{\|u' - v'\|^2}{|u_N - v_N|^2} = \frac{(\cos\theta(a_1 - b_1) - \sin\theta(a_N - b_N))^2 + (a_2 - b_2)^2 + \cdots + (a_{N-1} - b_{N-1})^2}{(\sin\theta(a_1 - b_1) + \cos\theta(a_N - b_N))^2}.$$

Finding the minimum of the last expression on $\{(a, b)/|a_N - b_N| \leqslant k\|a' - b'\|\}$ is equivalent to finding the minimum of $((X_1 \cos\theta - X_N \sin\theta)^2 + X_2^2 + \cdots + X_{N-1}^2)/(X_1 \sin\theta + X_N \cos\theta)^2$ on the set $\{X/X_1^2 + X_2^2 + \cdots + X_{N-1}^2 = 1, |X_N| \leqslant k\}$, or the minimum of $((X\cos\theta - Y\sin\theta)^2 + 1 - X^2)/(X\sin\theta + Y\cos\theta)^2$ on the set $\{(X, Y)/|X| \leqslant 1, |Y| \leqslant k\}$. It is attained when $X = 1$ and $Y = k$ which yields $\|u' - v'\|/|u_N - v_N| \geqslant k/(1 + 2k^2)$ and $|u_N - v_N| \leqslant ((1 + 2k^2)/k)\|u' - v'\|$.

Let CY_{e_1} be the open neighborhood of the origin defined in the y coordinates by $CY_{e_1} = B(0, r) \cap \{y/\|y'\| < \alpha\}$, where α is the constant of Corollary 2 (see Fig. 4).

LEMMA 4. *There exists a real-valued locally Lipschitzian function* ψ *defined in* $\mathbb{R}^{N-1}$ *such that in the* y *coordinates* $CY_{e_1} \cap \partial\Omega = \{y/y_N = \psi(y')\}$ *and* $CY_{e_1} \cap \Omega = \{y/y_N > \psi(y')\}$.

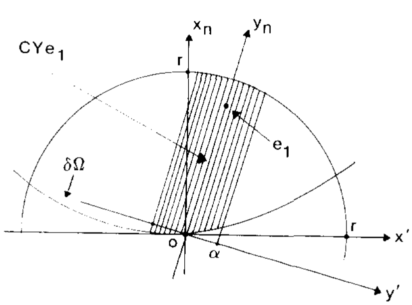

FIG. 4. The set CY_{e_1}.

Proof. Let y' satisfy $\|y'\| < \alpha$. From Corollary 2, the line through $(y', 0)$ parallel to e_1 (i.e., parallel to $0y_N$ in the y coordinates) intersects $\partial\Omega$ at only one point in $B(0, r)$; let (y', y_N) be that point. We put $y_N = \psi(y')$. Clearly ψ will satisfy the requirements if we can prove it is locally Lipschitzian in CY_{e_1}. Let u' and v' in $\mathbb{R}^{N-1}$ be such that $\|u'\| < \alpha$, $\|v'\| < \alpha$. $(u', \psi(u'))$ and $(v', \psi(v'))$ belong to $\partial\Omega$ and then can be written $(a', \varphi(a'))$, $(b', \varphi(b'))$ in the x coordinates and we have $|\varphi(a') - \varphi(b')| \leqslant k\|a' - b'\|$. Then from Lemmas 3, $|\psi(u') - \psi(v')| \leqslant ((1 + 2k^2)/k)\|u' - v'\|$. Therefore ψ is locally Lipschitzian.

LEMMA 5. *For any polynomial $P \in H_n$ we have*

$$\|(\partial/\partial y_N) P\|_{L^p(CY_{e_1})} \leqslant 4C(p)\, r^{-1} n^2 \|P\|_{L^p(CY_{e_1})}.$$

Proof. Let us consider CY_{e_1} as an union of segments parallel to $0y_N$ of length greater than $r/2$, bounded by surfaces $y_N = \psi(y')$ and $y_N = (r^2 - y_1^2 - \cdots - y_{N-1}^2)^{1/2}$. Since ψ is measurable (it is locally Lipschitzian, therefore continuous), using (4) we can write, when $\|y'\| < \alpha$

$$\int_{\psi(y')}^{(r^2 - y_1^2 - \cdots - y_{N-1}^2)^{1/2}} |(\partial/\partial y_N) P(y_1, \ldots, y_N)|^p \, dy_N$$

$$\leqslant (4C(p))^p\, r^{-p} n^{2p} \int_{\psi(y')}^{(r^2 - y_1^2 - \cdots - y_{N-1}^2)^{1/2}} |P(y_1, \ldots, y_N)|^p \, dy_N.$$

Integrating the two sides of this estimate over $\{y'/\|y'\| < \alpha\}$ leads to the required inequality.

To complete the proof of the main theorem, we set

$$e_2 = (0, 1, 0, 0, \ldots, 0, 2k), \qquad e_3 = (0, 0, 1, 0, \ldots, 0, 2k), \qquad \ldots,$$
$$e_{N-1} = (0, 0, 0, 0, \ldots, 0, 1, 2k), \qquad e_N = (-1, 0, 0, 0, \ldots, 0, 2k).$$

For every i, using a process similar to the one we used for e_1, we can find a neighborhood CY_{e_i} of the origin, such that for any $P \in H_n$,

$$\|D_{e_i} P\|_{L^p(CY_{e_i} \cap \Omega)} \leqslant 4C(p)\, r^{-1} n^2 \|P\|_{L^p(CY_{e_1} \cap \Omega)}.$$

We set $V_a = \bigcap_{i=0}^{N} CY_{e_i}$. Then for every i we have

$$\|D_{e_i} P\|_{L^p(V_a \cap \Omega)} \leqslant 4C(p)\, r^{-1} n^2 \|P\|_{L^p(\Omega)},$$

which completes the proof of inequalities (3).

To establish the (ii)-statement of the theorem we proceed in the same way using inequality (5) of Corollary 1 instead of inequality (4).

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